Electromagnetic Perturbations on πNN and πNN^* Couplings in the Chew-Low Model: Detailed Calculation*†

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Electromagnetic perturbations on πNN and πNN^* couplings are studied in the N-N^* reciprocal bootstrap model. We perform explicit calculations for both linear and curved D functions. The linear-D results agree with the conclusions reached in the previous paper. No enhancement of eigenvectors with eigenvalue equal to (or near to) $+1$ occurs, and thus our results are not accurate numerically. We do expect, however, that the prediction of very small effects ($\delta R/R \cong 1\%$) is reliable.

I. INTRODUCTION

IN this paper we perform a calculation of πN P-wave residue shifts, using the Dashen-Frautschi S-matrix method.[1] We obtain the customary equations[1]

$$\delta\mathbf{R}=A^{RR}\delta\mathbf{R}+A^{RM}(\delta\mathbf{m}/m_0)+\mathbf{D}, \quad (1.1)$$

and in our calculation we include the self-consistent ("bootstrap") terms as well as all driving terms in the low-energy region.

We discover that for linear D functions, A^{RR} has three eigenvalues equal to $+1$ and two eigenvalues equal to -1, while the other eigenvalues lie between $+1$ and -1. We also find that the driving terms are orthogonal to the eigenvectors with eigenvalue $+1$, so that no enhancement of these eigenvectors occurs. These results agree with the conclusions of the previous paper,[2] where it was shown that this type of behavior must occur. Qualitatively similar behavior is observed for curved D functions.

Since the driving terms are approximately orthogonal to the eigenvectors with eigenvalue near 1 (i.e., "no enhancement"), there is no single dominant pattern, and thus our numerical results are sensitive to details which are not well known. Thus our numerical results are not reliable, but we expect that the theory does at least produce reliable order-of-magnitude estimates. Our numerical results are small (partly as a consequence of no enhancement); we predict πN P-wave residue shifts of less than 1% of the unperturbed values. Experimentally, there are no measurements precise enough to compare with predictions as small as ours, though it can at least be said that there is no definite evidence for much larger residue shifts.

The plan of this paper is as follows: Section II gives the Dashen-Frautschi dispersion relations used, as well as a discussion of D functions. This section is included for completeness and does not represent new material. Section III contains a discussion of the infrared divergence problem arising in this type of calculation. Sections IV–VII describe the explicit evaluation of the dispersion integrals, using as input both "bootstrap" and driving terms. In Sec. VIII we find an analysis of the results and the conclusion of *no enhancement*. The numerical results of the computation are found in Tables II–VII. In Sec. IX we have a discussion of experimental data, and in Sec. X we compare our calculation with related work.

II. BASIC FORMULAS; NOTATION

We shall write the Jth partial-wave amplitude in $\pi N \to \pi N$ scattering, with initial and final isospin states $|I_iI_z\rangle$ and $|I_fI_z\rangle$, as $T(I_iI_z \to I_fI_z; J)$. As in the previous paper,[2] we denote the unperturbed amplitude by T_0 and the perturbation by $\delta T = T - T_0$. Then the first-order shift δR in the residue of an s-channel pole with mass M', isospin I_i, and spin J is given by[1]

$$\delta R(I_iI_z \to I_iI_z; J) = \frac{1}{2\pi i}\int_C \frac{D_{I_iJ}{}^2(W')}{(W'-M')^2}[1-D_{I_iJ}{}'(M')(W'-M')] \times \delta T(I_iI_z \to I_iI_z; J)dW', \quad I_i = I_f \quad (2.1)$$

$$\delta R(I_iI_z \to I_fI_z; J) = \frac{1}{2\pi i}\int_C \frac{D_{I_iJ}(W')D_{I_fJ}(W')}{W'-M'}\delta T(I_iI_z \to I_fI_z; J)dW', \quad I_i \neq I_f, \quad (2.2)$$

where D_{11} and D_{33} have zeros at the nucleon and N^* masses, respectively [we take $D_{11}'(M)=D_{33}'(M^*)=D_{31}(M)=D_{13}(M^*)=1$]. W is the center-of-mass energy, and the contour runs in a clockwise direction around all singularities except the initial direct-channel pole. From (2.1) and (2.2) we easily obtain the standard form[1]

$$\delta\mathbf{R}=A^{RR}\delta\mathbf{R}+A^{RM}(\delta\mathbf{m}/m_0)+\mathbf{D}, \quad (1.1)$$

where $\delta\mathbf{R}$ is a vector containing the residue shifts, $\delta\mathbf{m}/m_0$ is a mass-shift vector, A^{RR} and A^{RM} are matrices, and $\mathbf{D}$ is a "driving vector." We shall use the familiar

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[1] R. Dashen and S. Frautschi, Phys. Rev. 137, B1318 (1965).
[2] P. Babu, S. Frautschi, and N. Thornber, preceding paper, Phys. Rev. 172, 1389 (1968).

TABLE I. Linear combinations of residue shifts used in the analysis.

$$\delta R_{I=0,1}(\tfrac{1}{2}\to\tfrac{1}{2};\ J=\tfrac{1}{2}^+)=(\sqrt{\tfrac{1}{2}})[\delta R(\tfrac{1}{2}\ \tfrac{1}{2}\to\tfrac{1}{2}\ \tfrac{1}{2};\ J=\tfrac{1}{2}^+)\pm\delta R(\tfrac{1}{2}-\tfrac{1}{2}\to\tfrac{1}{2}-\tfrac{1}{2};\ J=\tfrac{1}{2}^+)];$$

$$\delta R_{I=0,2}(\tfrac{3}{2}\to\tfrac{3}{2};\ J=\tfrac{3}{2}^+)=(\sqrt{\tfrac{1}{4}})[\delta R(\tfrac{3}{2}\ \tfrac{3}{2}\to\tfrac{3}{2}\ \tfrac{3}{2};\ J=\tfrac{3}{2}^+)\pm\delta R(\tfrac{3}{2}\ \tfrac{1}{2}\to\tfrac{3}{2}\ \tfrac{1}{2};\ J=\tfrac{3}{2}^+)\pm\delta R(\tfrac{3}{2}-\tfrac{1}{2}\to\tfrac{3}{2}-\tfrac{1}{2};\ J=\tfrac{3}{2}^+)+\delta R(\tfrac{3}{2}-\tfrac{3}{2}\to\tfrac{3}{2}-\tfrac{3}{2};\ J=\tfrac{3}{2}^+)];$$

$$\delta R_{I=1}(\tfrac{3}{2}\to\tfrac{3}{2};\ J=\tfrac{3}{2}^+)=(\sqrt{1/20})[3\delta R(\tfrac{3}{2}\ \tfrac{3}{2}\to\tfrac{3}{2}\ \tfrac{3}{2};\ J=\tfrac{3}{2}^+)+\delta R(\tfrac{3}{2}\ \tfrac{1}{2}\to\tfrac{3}{2}\ \tfrac{1}{2};\ J=\tfrac{3}{2}^+)-\delta R(\tfrac{3}{2}-\tfrac{1}{2}\to\tfrac{3}{2}-\tfrac{1}{2};\ J=\tfrac{3}{2}^+)-3\delta R(\tfrac{3}{2}-\tfrac{3}{2}\to\tfrac{3}{2}-\tfrac{3}{2};\ J=\tfrac{3}{2}^+)];$$

$$\delta R_{I=1,2}(\tfrac{3}{2}\to\tfrac{1}{2};\ J=\tfrac{1}{2}^+)=-(\sqrt{\tfrac{1}{2}})[\delta R(\tfrac{3}{2}\ \tfrac{1}{2}\to\tfrac{1}{2}\ \tfrac{1}{2};\ J=\tfrac{1}{2}^+)\pm\delta R(\tfrac{3}{2}-\tfrac{1}{2}\to\tfrac{1}{2}-\tfrac{1}{2};\ J=\tfrac{1}{2}^+)];$$

$$\delta R_{I=1,2}(\tfrac{3}{2}\to\tfrac{1}{2};\ J=\tfrac{3}{2}^+)=(\sqrt{\tfrac{1}{2}})[\delta R(\tfrac{3}{2}\ \tfrac{1}{2}\to\tfrac{1}{2}\ \tfrac{1}{2};\ J=\tfrac{3}{2}^+)\pm\delta R(\tfrac{3}{2}-\tfrac{1}{2}\to\tfrac{1}{2}-\tfrac{1}{2};\ J=\tfrac{3}{2}^+)].$$

basis in which A^{RR} is block-diagonal[1] (i.e., we choose linear combinations of δR's which transform according to definite isospin breaking); these vectors are listed in Table I.

The D functions used in the dispersion integrals may be chosen in any convenient manner[3]; the only properties of the D's that were used in deriving the dispersion integrals were that the D's have the appropriate zeros and that the resulting integrals converge. We shall make use of two sets of D functions:

$$\left.\begin{aligned} &D_{11}=W-M,\\ &D_{13}=1,\\ &D_{31}=1,\\ &D_{33}=W-M^*, \end{aligned}\right\}\quad\text{(linear } D\text{ functions)}\qquad(2.3)$$

and

$$\left.\begin{aligned} &D_{11}=(W-M)(M-W_0)/(W-W_0),\\ &D_{13}=1,\\ &D_{31}=1,\\ &D_{33}=(W-M^*)(M^*-W_0)/(W-W_0),\\ &W_0=7M/3. \end{aligned}\right\}\quad\text{(curved } D\text{ functions)}\qquad(2.4)$$

The first set is convenient for certain theoretical considerations (this set was used in the previous paper[2]), while the second set is more likely to cause the integrals to converge and is thus more reliable for numerical calculations.

In the spirit of past calculations, we keep only the contributions due to nearby cuts and neglect all distant singularities when performing the dispersion integrals. Before actually embarking on a calculation of the individual terms, however, we need to look at the infrared divergence problem (due to infrared virtual photons) associated with the δR's.

III. INFRARED DIVERGENCES

The scattering amplitude for $\pi N\to\pi N$ will contain an infrared multiplicative correction arising from virtual infrared photons.[4,5] We first look at the nonrelativistic case. Here one can write the scattered wave for a particle in a potential

$$V=V_{\text{short-range}}+V_{\text{Coul}}$$

as[6]

$$\psi_{\text{scattered}}\overset{(r\to\infty)}{\longrightarrow}\frac{e^{ikr}}{r}e^{-i\gamma\ln 2kr}[f_{\text{Coul}}(\theta)+f'(\theta)],$$

where $f_{\text{Coul}}(\theta)$ has no resonance poles, but $f'(\theta)$ may; γ is a constant proportional to e^2. If the unperturbed (no Coulomb force) problem has a scattering resonance at $W=M^*$, then $f'(\theta)$ is proportional to $(R_0+\Delta R)/(W-M^*)$ near the resonance (R_0 is the unperturbed residue, while ΔR is a finite correction term). Thus

$$\psi_{\text{scattered}}\underset{r\to\infty;\,W\cong M^*}{\propto}\frac{e^{ikr}}{r}e^{-i\gamma\ln 2kr}\frac{R_0+\Delta R}{W-M^*};$$

expanding to first order in e^2, we obtain

$$\psi_{\text{scattered}}\underset{r\to\infty;\,W\cong M^*}{\propto}\frac{(1-i\gamma\ln 2kr)(R_0+\Delta R)}{W-M^*}.\qquad(3.1)$$

We note that the residue shift of the pole contains a finite piece (ΔR) as well as an infinite piece ($-R_0 i\gamma\ln 2kr$) arising from the infinite phase of the scattered wave.

Using the above example as a guide, we now examine the relativistic case. The total $\pi N\to\pi N$ scattering amplitude T is a product of a noninfrared piece ($T_0+\Delta T$) and an infrared piece arising from virtual photons.[4,5] To first order in α,

$$T=T_0+\Delta T+\sum_i \alpha T_0{}^i B^i$$

(the sum goes over linear combinations of $\pi N\to\pi N$ amplitudes), where B^i includes the contributions of virtual infrared photons. Taking the P-wave projections of T according to the standard projection formulas,[7] and letting $W\cong M^*$ (the resonance mass), we obtain

$$T\cong\frac{R_0+\Delta R+\frac{1}{4}\alpha R_0\int_{-1}^{+1}dx\,(3x^2+1)B(s,t)\,|_{W=M^*}}{W-M_0^*}+\frac{R_0\delta M^*}{(W-M_0^*)^2},\quad\text{near } W\cong M^*.\qquad(3.2)$$

[3] R. Dashen, Y. Dothan, S. Frautschi, and D. Sharp, Phys. Rev. 151, 1127 (1966).

[4] D. Yennie, S. Frautschi, and H. Suura, Ann. Phys. (N. Y.) 13, 379 (1961).

[5] J. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Co., Inc., Cambridge, Mass., 1955).

[6] A. Messiah, *Quantum Mechanics* (Interscience Publishers, Inc., New York, 1961), Vol. I, p. 430.

[7] S. Frautschi and J. Walecka, Phys. Rev. 120, 1486 (1960).

Here, ΔR is a finite correction term, whereas the integral involving B contains an infrared divergence. (The reader will recall that the analogous situation held in the nonrelativistic example above.)

Of course, the separation into finite and infinite pieces is not unique. If we write $B(s,t)=a(s,t)\ln[\lambda/b(s,t)]$, where λ is a fictitious photon mass, then $b(s,t)$ is not uniquely determined. [$a(s,t)$ is uniquely defined, however, and can be found by explicitly computing the divergent part of the relevant Feynman diagrams.] For convenience we choose $b(s,t)=\text{const}=m_\rho$. With this choice of B, Eq. (3.2) takes the form

$$T\cong\frac{R_0+\Delta R+d(M')\ln(\lambda/m_\rho)}{W-M_0'}+\frac{R_0\delta M'}{(W-M_0')^2},\qquad W\cong M', \tag{3.3}$$

where

$$M'=\begin{pmatrix}M^*\\M\end{pmatrix}$$

and

$$d(M')=\tfrac{1}{4}\alpha R_0\int_{-1}^{+1}dx\begin{pmatrix}3x^2+1\\1\end{pmatrix}a(s,t)\,|_{W=M'},\qquad \begin{pmatrix}J=\frac{3}{2}\\J=\frac{1}{2}\end{pmatrix}. \tag{3.4}$$

We shall use the Dashen-Frautschi dispersion integrals to calculate ΔR; the $\ln(\lambda/m_\rho)$ piece (if desired explicitly) could be obtained directly from the Feynman diagrams.

We now show which integrals to use in finding ΔR. The dispersion integrals (2.1) and (2.2) hold true for any amplitude δT that has a pole at the appropriate mass.[8] Defining

$$\delta g(W)=(T-T_0)-\frac{\text{“}T_0\text{”}}{\text{“}R_0\text{”}}d\begin{pmatrix}M^*\\M\end{pmatrix}\ln(\lambda/m_\rho),\qquad \begin{pmatrix}J=\frac{3}{2}+\\J=\frac{1}{2}+\end{pmatrix}, \tag{3.5}$$

where “T_0” and “R_0” refer to the T_0 with $I_i=I_f=J$, we see that for $J=\frac{3}{2}+$, for example,

$$\delta g(W)_{(W\cong M^*)}\cong\frac{[\Delta R+d(M^*)\ln(\lambda/m_\rho)]-d(M^*)\ln(\lambda/m_\rho)}{W-M_0^*}+\frac{R_0\delta M^*}{(W-M_0^*)^2}=\frac{\Delta R}{W-M_0^*}+\frac{R_0\delta M^*}{(W-M_0^*)^2}.$$

[8] R. Dashen and S. Frautschi, Phys. Rev. **135**, B1190 (1964).

Thus the dispersion integral for ΔR is (letting $I_i=I_f=\frac{3}{2}$)

$$\begin{aligned}\Delta R&=\frac{1}{2\pi i}\int_C\frac{D_{33}^2(W)\delta g(W)}{(W-M_0^*)^2}\times[1-D_{33}'(M^*)(W-M^*)]dW,\\ \Delta R&=\frac{1}{2\pi i}\int_C\frac{D_{33}^2(W)(T-T_0)}{(W-M_0^*)^2}\times[1-D_{33}'(M^*)(W-M^*)]dW-\frac{d(M^*)}{R_0 2\pi i}\left(\ln\frac{\lambda}{m_\rho}\right)\\ &\quad\times\int_C\frac{D_{33}^2(W)T_0}{(W-M_0^*)^2}[1-D_{33}'(M^*)(W-M^*)]dW.\end{aligned} \tag{3.6}$$

The last integral is evaluated *before* expanding the contour and gives trivially

$$-d(M^*)\ln(\lambda/m_\rho).$$

The other integral on the right-hand side of (3.6) equals $R_{\text{total}}-R_0=\delta R$. This piece will be found by expanding the contour in the usual fashion. Thus

$$\Delta R=\delta R-d(M^*)\ln(\lambda/m_\rho). \tag{3.7}$$

Now since ΔR is finite [(3.2) and (3.3)], an exact evaluation of the integrals for ΔR should give a finite result. In evaluating the first integral approximately, however, a spurious infrared divergence, similar to the spurious infrared divergence of mass-shift calculations,[9] may arise. We shall use the prescription of setting $\ln\lambda=\ln m_\rho$ in each of the spurious $\ln\lambda$ terms in ΔR. There has been some discussion in the literature about possible prescriptions to use[8–10]; we expect that the prescription $\ln\lambda=\ln m_\rho$ will give the correct order-of-magnitude results, although the detailed values of the residue shifts—even their signs—are questionable. Thus we obtain

$$\Delta R=\delta R|_{\ln\lambda=\ln m_\rho}+0, \tag{3.8}$$

and the full residue shift is given by

$$\Delta R+d(M')\ln(\lambda/m_\rho)=\delta R|_{\ln\lambda=\ln m_\rho}+d(M')\ln(\lambda/m_\rho). \tag{3.9}$$

Here $d(M')$ is given by (3.4), and in accordance with the nonrelativistic example at the beginning of this section, the residue shift has a finite piece [$\delta R|_{\ln\lambda=\ln m_\rho}$] and a divergent piece [$d(M')\ln(\lambda/m_\rho)$] arising from infrared photons. (The cross section, on the other hand, will of course be finite since the infrared divergences from virtual and real photons cancel.)

We would like to make the observation that the relation between the residue shifts considered and actual coupling constants is muddled because one does not separate the pole term from an overlapping cut (arising

[9] R. Dashen, Phys. Rev. **135**, B1196 (1964).
[10] G. Barton, Phys. Rev. **146**, 1149 (1966).

as $\lambda \to 0$). Thus in a sense both we and the experimentalist work with a pole and an overlapping cut (although theory in general could separate these two terms), and we discover that in our calculation one cannot set the residue proportional to the product of two coupling constants. (Explicitly, virtual photons connecting initial and final lines give rise to a branch cut of a box diagram which overlaps the pole term as $\lambda \to 0$.)

In the following sections we shall calculate the residue shifts ΔR $(=\delta R|_{\ln\lambda=\ln m_\rho})$ via the Dashen-Frautschi dispersion integrals (2.1) and (2.2). In other words, we shall simply find δR and then set each $\ln\lambda=\ln m_\rho$.

IV. A^{RR} AND A^{RM}

In this section we discuss the matrices A^{RR} and A^{RM}. The derivation of A^{RR} for linear D functions was given in the previous paper,[2] where it was shown that A^{RR} is the crossing matrix; the derivation for curved D functions is quite similar (there will now be additional nonlinear functions in the integrand). We find that A^{RR} for linear D has one eigenvalue equal to 1 for each I ($I=0, 1, 2$), while A^{RR} for curved D has one eigenvalue near to 1 for each I ($I=0$, 1, 2). The results for linear D confirm the conclusions of the previous paper[2] that here at least two eigenvalues must equal $+1$, at least two must equal -1, and that each eigenvalue must satisfy $|\lambda_i| \leq 1$. The eigenvalues of A^{RR} are listed in Table II.

To calculate A^{RM}, we make a distinction between external mass shifts (arising from external lines) and exchanged mass shifts (arising from internal lines). The effect of exchanged mass shifts is easy to find. We simply cross from the s to the u channel, expand about the pole terms, and do the integrals. The results are found in Table III and depend strongly on the choice of D functions.

To find the contributions from external mass shifts, more elaborate methods are necessary. Mass invariance[1] and group-theoretic arguments,[1] as well as direct examination of the dispersion integrals, yield the results shown in Table IV. These results are highly dependent on the D functions used. We note (as in the previous paper) that for linear D functions the mass shifts give no contribution to our residue shifts and will thus not contribute to any enhancement of eigenvectors of A^{RR} with eigenvalue equal to $+1$.

TABLE II. Eigenvalues λ_i of A^{RR}.

	(a) Linear D functions
$I=0$	$\lambda=1, -7/9$
$I=1$	$\lambda=1, 5/9, -7/9, -1$
$I=2$	$\lambda=1, -\frac{1}{5}, -1$
($I=3$	$\lambda=\frac{1}{3}$)
	(b) Curved D functions
$I=0$	$\lambda=0.95, -0.74$
$I=1$	$\lambda=0.85, 0.45, -0.67, -0.91$
$I=2$	$\lambda=0.81, -0.08, -0.87$
($I=3$	$\lambda=0.18$)

V. ONE-PHOTON EXCHANGE

The $\ln\lambda$ term (λ is a fictitious photon mass) arising from this diagram will be treated as described in the section on infrared divergences. Since the numerical result depends strongly on the prescription used to eliminate this spurious divergence (as in the case of the neutron-proton mass difference), we expect our answers to be of the correct order of magnitude, but doubtful as to sign and exact numerical value. In doing the integration, we use one-pole form factors (derived from the two-pole results of Hand, Miller, and Wilson[11]), and we neglect the isoscalar anomalous magnetic moment of the nucleon. These form factors are expected to be sufficient in our static-limit calculation. We set

$$\begin{aligned} F_1^p+F_1^n &= m_1^2/(m_1^2-t), \\ \kappa^p F_2^p+\kappa_n F_2^n &= 0, \\ (F_1^p+\kappa^p F_2^p)-(F_1^n+\kappa^n F_2^n) &= 4.70 m_3^2/(m_3^2-t), \\ \kappa^p F_2^p-\kappa^n F_2^n &= 3.70 m_4^2/(m_4^2-t). \end{aligned} \tag{5.1}$$

[We also use $F_\pi = m_\rho^2/(m_\rho^2-t)$.] Here $t=-2q^2 \times(1-\cos\theta_{qq'})$, where $\theta_{qq'}$ is the center-of-mass scattering angle; κ^p and κ^n are the nucleon anomalous magnetic moments ($\kappa^p=1.79$ and $\kappa^n=-1.91$ Bohr magnetons); F_1 and F_2 are the (Dirac and Pauli) nucleon form factors; and F_π is the pion form factor. We set $m_i^2=20m_\pi^2$ ($i=1,3,4$) and $m_\rho \cong 763$ MeV.

Using the familiar projection formula[7] for partial-wave amplitudes

$$\begin{aligned} \delta A\left[\pi_1 N_1 \to \pi_2 N_2;\ J=\begin{pmatrix}\frac{1}{2}^+\\ \frac{3}{2}^+\end{pmatrix}\right] \\ = \frac{1}{32\pi}\left\{\frac{(W+M)^2-\mu^2}{(W-M)^2-\mu^2}[A_1+(W-M)B_1]\right. \\ \left.+\left[-\begin{pmatrix}A_0\\ A_2\end{pmatrix}+(W+M)\begin{pmatrix}B_0\\ B_2\end{pmatrix}\right]\right\} \end{aligned} \tag{5.2}$$

TABLE III. Contributions of exchanged masses to residue shifts. (a) Linear D functions. (b) Curved D functions.

(a) All exchanged mass contributions equal zero. (b)	$A_I^{R,M^*\text{exch}}$	$A_I^{R,M\text{exch}}$
$\delta R_{I=0}(\frac{1}{2} \to \frac{1}{2};\ J=\frac{1}{2}^+)$	1.79α	0.00α
$\delta R_{I=0}(\frac{3}{2} \to \frac{3}{2};\ J=\frac{3}{2}^+)$	0.24α	1.96α
$\delta R_{I=1}(\frac{1}{2} \to \frac{1}{2};\ J=\frac{1}{2}^+)$	-1.33α	0.00α
$\delta R_{I=1}(\frac{3}{2} \to \frac{1}{2};\ J=\frac{3}{2}^+)$	2.36α	-0.87α
$\delta R_{I=1}(\frac{3}{2} \to \frac{3}{2};\ J=\frac{3}{2}^+)$	-0.44α	-1.31α
$\delta R_{I=2}(\frac{3}{2} \to \frac{3}{2};\ J=\frac{3}{2}^+)$	-0.08α	-1.46α
$\delta R_{I=2}(\frac{3}{2} \to \frac{1}{2};\ J=\frac{1}{2}^+)$	-3.16α	...
$\delta R_{I=2}(\frac{3}{2} \to \frac{1}{2};\ J=\frac{3}{2}^+)$	0.59α	...
$\delta R_{I=2}(\frac{3}{2} \to \frac{3}{2};\ J=\frac{3}{2}^+)$	-0.24α	...

[11] L. Hand, D. Miller, and R. Wilson, Rev. Mod. Phys. **35**, 335 (1963).

and taking the static limit $[q^2=(W-M)^2-\mu^2]$ for $A_{0,2}$ to eliminate contributions from cuts near $W=-M$,[9] we expand the contour to simplify the calculation. The results are presented in Table V and depend strongly on D as well as λ. For linear D and fixed I, the ratios of the results depend only on Clebsch-Gordan coefficients and are independent of the values of the anomalous magnetic moments, the positions of the simple poles in the form factors, and the value of λ (as long as λ is the same for each δR with the same I value). Explicit computation confirms the result of the previous paper[2] that one-photon exchange does not contribute to eigenvectors of A^{RR} with eigenvalue $+1$.

t-channel exchanges other than one-photon exchange will be explicitly neglected in our static-model calculation.

VI. RIGHT-HAND CUT

We need to evaluate that part of the driving term **D** coming from the right-hand cut [i.e., the piece corresponding to $-\int_M^\infty \delta R_\alpha'(W)dW$ in (2.9) of the previous paper[2]]. Unitarity of the S matrix gives us an expression for ImδT involving phase shifts and photoproduction multipoles (we look only at the πN and γN intermediate states and neglect such processes as $\pi N\gamma$, $N^*\gamma$, and $\pi\pi N$ intermediate states).

Now the explicit D functions we have been using (linear D and curved D) have no phases along the right-hand cut. Thus we would set $\mathrm{Im}D_iD_j\delta T=D_iD_j\mathrm{Im}\delta T$ along this cut. But unitarity only gives us ImδT in terms of ReδT, and we have no model with which to make an independent estimate of ReδT. Thus we do not have any way to find numerical values for ImδT. This situation is described in terms of δ (phase shift) in an appendix of Ref. 3. To obtain an estimate of $\mathrm{Im}D_iD_j\delta T$, we shall set the phase of each D function equal to minus the real part of the scattering phase shift. This is admittedly only an approximation and can only give us an idea of the order of magnitude of the terms involved. The approximation is not so bad for amplitudes with small phase shifts, but is doubtful whenever a (33) phase shift is involved.

TABLE IV. Contributions of external mass shifts to residue shifts. (a) Linear D functions. (b) Curved D functions.

(a) All external mass contributions equal zero. (b)	$A_I^{R,M^{\mathrm{ext}}}$	$A_I^{R,\mu^{\mathrm{ext}}}$
$\delta R_{I=0}(\frac{1}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	-2.53α	0α
$\delta R_{I=0}(\frac{3}{2}\to\frac{3}{2};\ J=\frac{3}{2}^+)$	-2.30α	0α
$\delta R_{I=1}(\frac{1}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	0.84α	...
$\delta R_{I=1}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	-4.70α	...
$\delta R_{I=1}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{3}{2}^+)$	2.36α	...
$\delta R_{I=1}(\frac{3}{2}\to\frac{3}{2};\ J=\frac{3}{2}^+)$	-1.71α	...
$\delta R_{I=2}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	...	0α
$\delta R_{I=2}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{3}{2}^+)$	...	0α
$\delta R_{I=2}(\frac{3}{2}\to\frac{3}{2};\ J=\frac{3}{2}^+)$	...	0α

TABLE V. Contributions from one-photon exchange.

	Linear D functions[a]	Curved D functions
$\delta R_{I=0}(\frac{1}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	$(\frac{2}{3}\sqrt{2})\alpha(S_0-\frac{1}{6}Q_0)$	$\alpha[-0.93\ln(M/\lambda)+0.76]$
$\delta R_{I=0}(\frac{3}{2}\to\frac{3}{2};\ J=\frac{3}{2}^+)$	$-\frac{2}{3}\alpha(S_0-\frac{1}{6}Q_0)$	$-\alpha[-0.97\ln(M/\lambda)+0.74]$
$\delta R_{I=1}(\frac{1}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	$-(\frac{2}{3}\sqrt{2})\alpha(P_0)$	$-\alpha[-0.64\ln(M/\lambda)+0.28]$
$\delta R_{I=1}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	$\frac{2}{3}\alpha(P_0)$	$\alpha[-1.43\ln(M/\lambda)+1.31]$
$\delta R_{I=1}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{3}{2}^+)$	$-\frac{2}{3}\alpha(P_0)$	$-\alpha[-1.07\ln(M/\lambda)+1.08]$
$\delta R_{I=1}(\frac{3}{2}\to\frac{3}{2};\ J=\frac{3}{2}^+)$	$-\frac{1}{3}(\sqrt{5})\alpha(P_0)$	$-\alpha[-1.77\ln(M/\lambda)+1.39]$
$\delta R_{I=2}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	$-\frac{2}{3}\alpha(S_0-\frac{1}{6}Q_0)$	$-\alpha[-1.58\ln(M/\lambda)+1.09]$
$\delta R_{I=2}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{3}{2}^+)$	$\frac{2}{3}\alpha(S_0-\frac{1}{6}Q_0)$	$\alpha[-1.18\ln(M/\lambda)+1.31]$
$\delta R_{I=2}(\frac{3}{2}\to\frac{3}{2};\ J=\frac{3}{2}^+)$	$-\frac{4}{3}\alpha(S_0-\frac{1}{6}Q_0)$	$-\alpha[-1.93\ln(M/\lambda)+1.48]$

[a] $S_0=4.70P_0$; $Q_0=3.70P_0$; $P_0=\mathrm{m}_\rho{}^2m_1{}^2[A\ln\lambda+B\ln m_\rho+C\ln m_1]$, where $A=(\mathrm{m}_\rho{}^2m_1{}^2)^{-1}$, $B=[m_\rho{}^2(m_\rho{}^2-m_1{}^2)]^{-1}$, and $C=[m_1{}^2(m_1{}^2-m_\rho{}^2)]^{-1}$. Thus $P_0=[-\ln(M/\lambda)+0.82]$.

For the photoproduction multipoles, we simply use the largest (N^* resonance and Born part of $M_{1\pm}$ from the e amplitudes) parts of the CGLN[12] multipoles. These multipoles are expected to be valid up to the $N^*(1236)$ resonance region. We set all absorption parameters (due to complex phase shifts) equal to 1, which is a good approximation for $T_\pi\leq 300$ MeV. πN phase shifts are obtained from the analysis of Roper *et al.*[13]

The results (which do not depend on which of the two D functions was used) are listed in Table VI. One can gain qualitative understanding by noting that for residue shifts with $I=1$, one isoscalar vertex and one isovector vertex are needed,[9] whereas for $I=0$ or 2 no isoscalar vertex is required. The available isoscalar vertex (nucleon spin flip) is small, and hence the $I=1$ contributions from the γN intermediate state are expected to be small.[9] For $I=0$ and $I=2$ larger contributions are expected, especially for $I_i=I_f=\frac{3}{2}$, since the photoproduction amplitude here is large due to N^* resonance effects.

VII. LEFT-HAND CUT—RELATION TO THE RIGHT-HAND CUT

The contributions from the left-hand cut (other than N and N^* exchange) are directly related in the static limit to the contributions on the right-hand cut via the matrix A^{RR}. This was explicitly shown in the pre-

TABLE VI. Contributions due to the existence of the γN direct-channel intermediate state.

$\delta R_{I=0}(\frac{1}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	0.72α
$\delta R_{I=0}(\frac{3}{2}\to\frac{3}{2};\ J=\frac{3}{2}^+)$	3.35α
$\delta R_{I=1}(\frac{1}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	-0.35α
$\delta R_{I=1}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	0.13α
$\delta R_{I=1}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{3}{2}^+)$	-0.22α
$\delta R_{I=1}(\frac{3}{2}\to\frac{3}{2};\ J=\frac{3}{2}^+)$	0.00α
$\delta R_{I=2}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	-0.51α
$\delta R_{I=2}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{3}{2}^+)$	0.66α
$\delta R_{I=2}(\frac{3}{2}\to\frac{3}{2};\ J=\frac{3}{2}^+)$	-3.35α

[12] G. Chew, M. Goldberger, F. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

[13] L. Roper, B. Wright, and B. Feld, Phys. Rev. **138**, B190 (1965).

vious paper[2] for the linear-D case [Eq. (4.2)[2]]:

$$[\text{left-hand cut}]_\alpha = -(A^{RR})_{\alpha\beta}[\text{right-hand cut}]_\beta, \quad (7.1)$$

where for linear D the matrix A^{RR} equals the crossing matrix X. We now show that (7.1) is approximately true for curved D as well as for linear D. The proof is as follows.

The contribution to δR_α (α represents I, I_i, I_f, and J) from the left-hand cut is

$$\delta R_\alpha \leftrightarrow \frac{1}{2\pi i}\int_{\mathcal{C}} f(W_s)\delta T_\alpha(W_s)dW_s, \quad (7.2)$$

where $f(W_s)=1$ for linear D and involves D functions for curved D. But

$$\delta T_\alpha(W_s) \cong X_{\alpha\beta}\delta T_\beta(W_u) \quad (7.3)$$

from crossing. Changing variables from W_s to W_u ($W_s \cong 2M - W_u$) in the integral, we obtain

$$\delta R_\alpha \overset{\sim}{\leftrightarrow} \frac{1}{2\pi i}\int_{\mathcal{C}} f(2M-W_u)X_{\alpha\beta}\delta T_\beta(W_u)(-dW_u),$$

where the cut starts at $W_u = M$ and runs to $+\infty$; i.e.,

$$\delta R_\alpha \overset{\sim}{\leftrightarrow} -X_{\alpha\beta}\frac{1}{2\pi i}\int_{\mathcal{C}} f(2M-W)\delta T_\beta(W)dW. \quad (7.4)$$

Making the approximation of evaluating $f(2M-W)$ at $W=M$ (when $J=\frac{1}{2}$) or at $W=M^*$ (when $J=\frac{3}{2}$), one obtains

$$\delta R_\alpha \overset{\sim}{\leftrightarrow} -(A^{RR})_{\alpha\beta}\frac{1}{2\pi i}\int_{\mathcal{C}} \delta T_\beta(W)dW$$

$$\cong -(A^{RR})_{\alpha\beta}[\text{right-hand cut piece}]_\beta. \quad (7.5)$$

Thus we have shown that if $\mathbf{D}_1$ is the contribution from the right-hand cut, then the same states in the u channel contribute approximately $-A^{RR}\mathbf{D}_1$ (in the static limit) to $\delta\mathbf{R}$, even for curved D functions.

Thus, as in the previous paper[2] for linear D, eigenvectors of A^{RR} with eigenvalue equal to (or near to) $+1$ receive no contribution from the combination $\mathbf{D}_1+(-A^{RR}\mathbf{D}_1)$; no enhancement occurs here.

VIII. ANALYSIS OF RESULTS—NO ENHANCEMENT

In the preceding sections we have found numerical values for the contributions from the nearby singularities, using first linear and then curved D functions. We need to use these data to solve for our residue shifts.

We have seen that

$$\delta\mathbf{R} \cong A^{RR}\delta\mathbf{R} + A^{RM}(\delta\mathbf{m}/m_0) + (I-A^{RR})\mathbf{D}_1 + \delta\mathbf{R}_t, \quad (8.1)$$

where $\mathbf{D}_1$ comes from the right-hand cut, and $\delta\mathbf{R}_t$ is approximated by one-photon exchange. It was found explicitly that A^{RR} (for linear D) has three eigenvalues equal to $+1$, but that the remaining terms on the right-hand side of (8.1) are orthogonal to the corresponding eigenvectors. These results agree with the general conclusions of the previous paper.[2]

Before actually solving for our δR's, we need to investigate the behavior of our equations under charge conjugation. In terms of coupling constants, we can write

$$\delta g(\pi^- p \to n) = \mp\delta g(\pi^+ n \to p), \quad C=\pm 1. \quad (8.2)$$

If residues are assumed proportional to products of coupling constants, then this condition becomes

$$\delta R_I(\tfrac{1}{2}\to\tfrac{1}{2};\ J=\tfrac{1}{2}^+) = \sqrt{2}\begin{pmatrix}1\\-2\end{pmatrix}\delta R_I(\tfrac{3}{2}\to\tfrac{1}{2};\ J=\tfrac{1}{2}^+), \quad I=1,\ C=\pm 1. \quad (8.3)$$

Forming the matrix P which leaves states of $C=+1$ invariant and projects out states with $C=-1$, we find that numerically $PA_{I=1}P$ is quite different from $A_{I=1}$. (P does not affect states with $I=0$ or 2.) Thus $A_{I=1}$ is not C invariant, and hence we expect the numerical results for $I=1$ to be especially unreliable. For $I=1$, we shall use $A_{I=1}$.

We can now solve for our residue shifts. For curved D the solution is straightforward. For linear D there are eigenvalues equal to $+1$; thus

$$\delta\mathbf{R}_I = \mathbf{D}_1 + a_I\mathbf{x}_h + \mathbf{x}_P \text{ (linear } D), \quad (8.4)$$

where $\mathbf{x}_h$ is a homogeneous solution $(I-A^{RR})\mathbf{x}_h=0$, $\mathbf{x}_P$ satisfies $(I-A_I{}^{RR})\mathbf{x}_P=\delta\mathbf{R}_t$, and a_I is a constant. a_I is determined by equating δR's for linear and curved D and taking the average of the a_I's thus obtained. All N^* mass shifts have been set equal to zero.

The final values for $\Delta\mathbf{R}$ are tabulated in Table VII. They do not depend so much on the D functions used,

TABLE VII. Residue shifts ΔR (neglecting N^* mass differences), obtained by evaluation of the N, N^*, γN, πN, and γ exchange diagrams and the πN and γN direct-channel intermediate states.

	Linear D functions[a]	Curved D functions
$\delta R_{I=0}(\frac{1}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	$\sqrt{2}a_{I=0}+2.2\alpha$	undetermined
$\delta R_{I=0}(\frac{3}{2}\to\frac{3}{2};\ J=\frac{3}{2}^+)$	$a_{I=0}-0.89\alpha$	undetermined
$\delta R_{I=1}(\frac{1}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	-2.9α	-2.7α
$\delta R_{I=1}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	4.6α	4.9α
$\delta R_{I=1}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{3}{2}^+)$	-2.9α	-2.9α
$\delta R_{I=1}(\frac{3}{2}\to\frac{3}{2};\ J=\frac{3}{2}^+)$	4.0α	4.1α
$\delta R_{I=2}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)$	-2.5α	-2.3α
$\delta R_{I=2}(\frac{3}{2}\to\frac{1}{2};\ J=\frac{3}{2}^+)$	2.1α	2.2α
$\delta R_{I=2}(\frac{3}{2}\to\frac{3}{2};\ J=\frac{3}{2}^+)$	-2.3α	-2.4α

[a] $a_{I=0}$ = undetermined const. $R_0(\frac{1}{2}\to\frac{1}{2};\ J=\frac{1}{2}^+)\cong -11.1$. $R_0(\frac{3}{2}\to\frac{3}{2};\ J=\frac{3}{2}^+)\cong -5.6$.

but are strongly dependent on the value of the photon mass λ. The residue shifts ΔR are seen to be approximately of the order of the fine-structure constant α. No enhancement of eigenvectors of A^{RR} with eigenvalue equal to or near to $+1$ occurs, in agreement with the results of the previous paper.[2] We believe that in spite of the uncertainties of the calculation, the conclusion of small residue shifts is probably qualitatively correct.

IX. EXPERIMENTAL DATA

Precise data on πN residue shifts are scarce. The experimental error bars on g^2 are usually fairly large, and no definite conclusions can be reached. Other (theoretical) work on charge independence indicates that the residue shifts should be small.

The most accurate determination of a πN residue comes from dispersion relations combined with pion-nucleon scattering results[14,15]; this yields a value of $f^2=0.081\pm0.0018$. NN scattering data give the values[16,17] $g^2=13.8\pm1.9$ and $g^2=14.7\pm0.9$. An examination of light nuclei[18,19] indicates that charge independence probably holds to within 1%. Studies on NN scattering lengths[20–22] indicate that coupling shifts of a few percent could explain the scattering-length discrepancies, but that other effects could be of equal importance. A study of β decay indicates[23] that the experimental results are consistent with a few percent of charge-independence violation.

When $J=\frac{3}{2}^+$, measurement of N^* parameters has yielded[24,25]

$$\Gamma(\tfrac{3}{2},\ \tfrac{3}{2})-\Gamma(\tfrac{3}{2},-\tfrac{1}{2})=0.4\pm3.1\ \text{MeV},$$
$$\Gamma(\tfrac{3}{2},-\tfrac{3}{2})-\Gamma(\tfrac{3}{2},\ \tfrac{3}{2})=25\pm23\ \text{MeV}.$$

Both of these results are consistent with small residue shifts, but the error bars are too large to permit numerical comparison with the theory.

Thus the experimental data do not give a definite value, but at least they do not decisively indicate larger residue shifts than our predictions.

X. COMPARISON WITH RELATED ANALYSES

Recent calculations by Dashen *et al.*[26] of coupling shifts in $SU(3)$ symmetry breaking have also included predictions for πNN and πNN^* coupling shifts. These calculations predict the $I=1$ part of the coupling shifts and approximate the driving terms by mass-shift contributions. Dashen *et al.*[26] obtain $I=1$ $\delta g/g$'s (πNN vertex) which are very small (on the order of 0.1α or less) and the estimate $\delta\Gamma(N^{*-}-N^{*++})\cong1$ MeV. Our $\delta R/R_0$'s are roughly of order α; this result agrees with the $\delta\Gamma$ estimate but is larger than the $(\delta g/g)_{\pi NN}$ results of Dashen *et al.*[26]

Some of the δg's have also been estimated and found to be small, by Goldberg.[27]

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